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Abstract

Social network formation models are often compared by their network structures, which satisfy specific equilibrium or welfare properties. Here, we concentrate on welfare criteria and define properties of utility function which are causal for certain network structures. We hope the identification of different properties of utility function will enhance the understanding of the relationship of different network formation models. If this line of research is continued, a kind of engineering of network formation models might arise such that actual social networks can be directly described by appropriate utility functions.

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1 Classification of Network Models

Manski (2000), in a remarkable article, tackles the state of the art of economic research on social interaction effects. The social interaction literature is closely related to social network topics and can be seen as a pre-amplifier of the social network literature.[1] While social interaction theory and empirical methods are based on aggregates like "peer influences," "neighborhood effects," and "social capital," the social network literature is based on individuals and their relationships. In a sense, we can state that the individual is the atomic module of economic thinking. However, there are several parallels between both branches of economics. Manski (2000) argues that

> "The weak state of empirical research on social interaction should be a matter of concern both to economists with a policy focus and those with a theoretical focus. For years, economists have speculated about the role of nonmarket interactions in determining such matters of public interest as schooling outcomes, employment patterns, participation in welfare programs, crime rates, and residential segregation. To inform policy, we need to replace speculation with sound empirical analysis. Economic theorists need to know what classes of social interactions are prevalent in the real world."[2]

Many arguments also apply to the social network literature. Applied working researchers are mainly sociologists and physicists, while economists seldom contribute to empirical research. One reason for this finding might be that especially successful applied working economists are often econometricians and not merely statisticians. Econometricians need not only statistical methods but also applicable theories which are based on sound economic reasoning. Hence, before econometricians can analyze the "classes of social interactions" as proposed by Manski, respective classes of actual social networks a classification scheme of social network theories might be the grounding where applied working researchers can work with. In recent years, economists developed several game theoretical models which improved our understanding of social networks. However, nearly all models invented are very stylized and typically point out only one specific argument. A complete theory which is applicable to empirical social network data seems far away. Theories developed by sociologists often characterize some features of actual social networks. In the short-run, this focus on existing objects is a clear advantage of

[1]Historically, this might be wrong since famous sociologists started to work on social networks in the thirties of the last century.

[2]Manski continues with a very fundamental statement "Otherwise, theory risks becoming only a self-contained exercise in mathematical logic."

sociologists, although, the descriptions made are not based on well-developed tools. Therefore, the advancement of sociological theories is restricted to the construction of new hypothesis.[3]

What was said about social network models in general also applies to social network formation models in particular. Network formation theories are among the hottest topics in the social network and the whole economic literature. The utility functions of this models are often not modelled to describe really existing social networks, but researchers often prefer to find full-fledged solutions. One of these network models constructed by economists is the symmetric connection model. This model, invented by Jackson and Wolinsky (1996), is one of the best-known network formation models available. It is a cornerstone of the social network literature due to the notation used, game-theoretical concepts, and its theoretical implications. Other often cited network formation models are the co-author model[4], the one-way and two-way flow models of Bala and Goyal (2000), job contact network of Calvó-Armengol (2004), the formation of risk-sharing networks of Bramoullé and Kranton (2005), etc. What is the relationship among all these models? Typically, network formation models are compared by their outcomes, i.e. the network structure of the equilibrium networks or efficient networks. A much more demanding task than the simple comparison of outcomes is the comparison of assumption sets. Sometimes it is said that regularities or asymmetric networks arise because of specific features of utility functions, but a full-fledged comparison of different characteristics of utility functions is not given. If this relationship between utility functions and outcomes is clearly worked out, a classification scheme of social network formation models seems in reach.

Such a classification scheme of social network formation models might be of great value to both theorists and applied researchers. While Jackson and Wolinsky (1996) introduce a specific form of the model, we characterize the whole class of symmetric connection models. We propose calling this class of network formation models the SCM-class. This class consists of many arbitrary models which we show below. It is important to emphasize that our classification scheme for the SCM-class is based on the utilitarian welfare property and not on an equilibrium concept. However, an extension and specialization of our classification scheme defined below might be possible. The paper is structured in the following way. In Section 2 we define the SCM-class. In Section 3 we derive the MSCM-class which is closely related to the SCM-class. In Section 4 we analyze rival networks and our conclusion in Section 5 summarizes the main results.

[3]It is important to note that some sociologists, Braun and Gautschi (2006), have recently started to use game-theoretical tools. This is a positive signal for the development towards a unified social network theory and might lead to the genesis of the science of social networks.

[4]Also introduced in Jackson and Wolinsky (1996).

2 The SCM-Class based on the Utilitarian Welfare

The most famous network formation model is without doubt the symmetric connection model. Since its publication, several extensions of the model have been analyzed. For instance, researcher papers written by Johnson and Gilles (2000), Carayol and Roux (2005), and also Bala and Goyal (2000) are grounded on the general ideas of the symmetric connection model. A major reason for the success of this formation model is its simplicity. We continue this tradition for exactly the reason mentioned and hope that further research is rewarded by classification schemes of more elaborated network models. The essential part of any game-theoretical model is the specification of the utility function. Given our research problem posed, we have to identify properties of utility functions such that a certain kind of network structures are most efficient. As a measure for efficiency we use the standard utilitarian welfare concept which sums up utility levels of all players. Thereby, we say a utility function consists of benefits and costs. The benefits produce positive utility, and costs negative utility. First, we define properties of utility functions and afterwards we are able to define the SCM-class. To indicate that the SCM-class is defined with respect to the utilitarian welfare measure, often abbreviated by W, we call our class the SCM(W)-class.

Definition 1 (Anonymity)
A utility function is said to be anonymous if given a network structure any permutation of players yields the same set of utility levels.

Anonymity fails if, for example, one player is productive if he is directly related to another specific player and produces nothing otherwise. Also, heterogenous player characteristics might lead to non-anonymous settings.

Definition 2 (No Indirect Link-Costs)
A utility function exhibits the no-indirect-link-cost property if costs are only paid for direct links.

The no-indirect-link-cost property is a very natural assumption since costs are only paid by the player who decides whether to form a link or not.

Definition 3 (Constant Costs)
The cost to form one link is independent of any other factor.

This is a very often assumed assumption which simplifies many computational tasks since it allows the definition of cost classes. Johnson and Gilles (2000) have shown how the results of the symmetric connection model change if the formation of costs is not constant. Given the author´s setup, for low enough costs the only efficient network formation is the line in the modified symmetric connection model.

Definition 4 (Constant Benefits)
The benefits out of direct and indirect links is independent of any other factor.

Given this assumption, many demanding computational tasks are redundant.

Definition 5 (Additive Separability of Costs)
A utility function exhibits additive separability of costs if the costs are additively separated from benefit variables.

This definition implies that marginal utility with respect to a small change in costs is independent of the benefit level. Furthermore, we define the following network characteristic.

Definition 6 (Additive Separability of Benefits)
A utility function exhibits additive separability of benefits if benefits out of direct and indirect links of any degree are additive separable.

In accordance with the additive separability of costs property this implies that marginal effects of the benefits out of direct and indirect links are easily identified.

Definition 7 (Distance Network)
In a distance network utility levels are only affected by the distance between two players.

Here, distance is defined as the number of links on the shortest path from one player to another player given a network. In a distance network the form of the network is only important because it determines the distances between all players, however, no other arguments which might be relevant for actual networks, like synergies, competition, etc. have an impact. A non-distance network is, for example, the job contact network model of Calvó-Armengol (2004). There not only the distance determines the network but also the number of rivals, i.e. the number of indirect links of distance two. To distinguish several forms of distance networks we define the "d-distance networks".

Definition 8 (d-Distance Network)
In a d-distance network utility levels are only affected by the distance between two players where the maximal distance between two players is d.

The original symmetric connection model is a ∞-distance network since indirect links of any distance can contribute to the utility function of players. Jackson and Wolinsky (1996) call a symmetric connection model one where $d < \infty$ truncated.[5]

Definition 9 (Strict Ranking of Benefits)
Given link formation is costless, a utility function exhibits the strict rank of link benefits if more distant links are less beneficial.

This property is closely related to the several properties mentioned since the comparison of benefits requires the identification of different beneficial effects. A sufficient condition allowing the comparison of this variable is the "additive separability of benefits". In many real world situations we can assume that this property is fulfilled. If players are more distant it seems reasonable to assume that in many situations less valuable information is exchanged. However, again this assumption might fail in reality if one player has an indirect link to a very productive player and this might produce a higher benefit than an direct link of the some player. All the definitions above are used to define the SCM(W)-class.

[5] See Proposition 3 in the original work.

Definition 10 (SCM(W)-class)
Let all variables of a network formation model be real and continuous. This model is one out of the SCM(W)-class if its utility function exhibits the following characteristics:

(i) ∞-Distance Network
(ii) Anonymity
(iii) Constant Costs
(iv) Constant Benefits
(v) Additive Separability of Costs
(vi) Additive Separability of Benefits
(vii) No Indirect Link-Costs
(viii) Strict Ranking of Benefits.

Given these definitions the following theorem holds.

Theorem 1 *If a network is one of the SCM(W)-class then the empty network, the star network, and the complete network are the only efficient networks.*

Proof: The continuity of variables and that all variables are real guarantees that all variables are well-defined. The most important definition is the "Distance Network" characteristic which allows simple comparisons of direct links, indirect links, and costs to form links.[6]

The anonymity property allows us to focus on the consideration of network structures. Hence, the position of a special individual player is of no importance for the utility functions and therefore has no influence on the efficiency measure. If the costs and benefits of forming links are constant, we have to consider only a restrictive number of cost and benefit variables in the model. Therefore, if the costs of forming links relative to their benefits are low then the complete network is formed since less distant links are more valuable.

Given additive separability of costs and benefits if the costs to form links rise then the utility out of direct links shrinks while the benefit of indirect links is due to our no indirect link-cost property independent of any costs. The additive separability properties also allow the direct comparison of direct and indirect links. Thus, if indirect links of distant two are more valuable than the net benefit of direct links then the indirect links of distant two produce the maximal

[6]For instance, the model of Johnson and Gilles (2000) also exhibits this property.

benefit among all links. This holds due to our assumption of strict ranking of benefits. This part of the proof, which shows that the star is among the class of efficient networks, is divided into three steps.

First, let us consider a network which consists of a minimally connected component. In a minimally connected component the number of direct links and the number of indirect links of any distance are independent of the network structure.[7] The number of indirect links of distance two are maximized in the star therefore, the star is the efficient network structure of all minimally connected components. Second, it is immediately seen that the star is also the most efficient network formation among all networks which consist of one component only. This holds since any other network formation reduces the number of indirect links of distance two. However, this reduces our welfare since the indirect links are the most beneficial. Third, let the network consist of more than one component where each component produces positive welfare. Adding some links such that at least two components form a new larger component is then a welfare improvement. Therefore, the efficient network consists of one component and the star is the only efficient network in the medium range.

If the costs are high then the empty network is the only efficient network. There is no other network formation which is efficient because the star is minimally connected. This implies that if direct links are very costly such that their maintenance is not compensated by the benefits produced by the indirect links, then the empty network is the only solution to the maximization problem. □

Given the definition of the SCM(W)-class we can directly derive the following proposition about the set of efficient networks.

Proposition 1 *Suppose the cost of forming links is high such that the empty network is efficient then in any network formation model of the SCM(W)-class there is some higher number of players such that the star becomes efficient.*

Proof: Given theorem 1 and supposing the empty network is efficient given a fixed number of players and a constant cost level, then the welfare of the star is negative since the welfare of the empty network is zero. However, the welfare of the star depends positively on the number of players while the welfare of the empty network is independent of the number of players. The welfare of the star depends positively on the number of players since even if direct links produce negative utility levels the benefit of indirect links of distance two is always greater than zero.

[7] If a network consists of N players and it is minimally connected then there are $N-1$ direct links and each player has $N-2$ indirect links. This fixes the total number of links of any distance.

The number of direct links increases linearly in the star network while the number of indirect links is a polynomial function of degree two. This holds because adding a player to a star increases the number of indirect links of both all incumbent players and the new player. Hence, eventually the empty network is no longer efficient and the star becomes efficient if the number of players is increased repeatedly. □

The correctness of the proof is also seen in Jackson and Wolinsky (1996), where the lower bound for the star network is independent of the number of players while the upper bound depends positively on the number of players in the network. That the example in Jackson and Wolinsky (1996) is not the only existing network formation model of the SCM(W)-class is shown in the next Section where we introduce another very simple network formation model. It is also possible to show that the SCM(W)-class contains arbitrary many network formation models.

The modification of some assumptions used in the definition 10 can lead to other (sub-)classes of network formation models. Hence, given theorem 1 the following propositions also applies.

Proposition 2 *Suppose a utility function, i.e. a class of network formation models, is characterized by property (ii) to (viii) in definition 10 and property (i) is replaced by (i') d-distance network, then the set of efficient networks consists of the empty network, the star network, and the complete network if* $d \geq 2$.

Proof: This holds since the most beneficial indirect link is the link of distance two. Hence, as long as these indirect links contribute given the strict ranking of benefits no other network formation can be efficient. □

If only direct links contribute, i.e. 1-distance networks are considered, then indirect links are not beneficial. Of course, there are no constellations where indirect links are more beneficial than direct links. This excludes that the set of efficient networks contains the star network. This argument proves the following Lemma.

Lemma 1 *If a utility function satisfies property (ii) to (viii) in definition 10 and also satisfies the* 1*-distance network property, then the set of efficient networks contains only the empty network and the complete network.*

If we relax the "strict ranking of benefits" assumption instead of the "*d*-distance network" characteristic, we can derive sets which contain new efficient network formations. For a certain assumption set, the symmetric double star is among the set of efficient network structures. Therefore, we define this network formation.

Definition 11 *If the number of players is even a symmetric double star consists of two stars of identical size where only the center of both stars are connected.*

Such a double star has a minimal number of links and maximizes the number of links of distance three. If the number of players is even, this holds due to the symmetry of the double star since an asymmetric double star[8] has more links of distant two and, therefore, less links of distant three. If the number of players is odd, then the sizes of both stars of the symmetric double star differ by one player. This network formation is used for our next result.

Proposition 3 *If all variables are real and continuous, a utility function satisfies property (i) to (vii) and the strict ranking of benefits property is changed to* $b(1) > b(3) > b(2) > b(4) > b(5) > b(6) > \cdots > b(\infty)$ *where* $b(d)$ *is the benefit of distance* d *in a* ∞*-distance network*[9] *then among the set of efficient networks is the empty network, the symmetric double star, and the complete network.*

Sketch of the proof: The proof is similar to the one given in theorem 1. The only difference is that the star is replaced by the double star. If the costs are in a medium range, then the double star becomes efficient. This holds since we need a minimal connected network to maximize the number of indirect links. Given a minimal connected network, the number of indirect links is constant. To maximize the number of links of distance three among all indirect links the double star must be formed. □

The statement of proposition 1 is also applicable to this proposition. Hence, if a parameter combination in a specific network formation model is given such that the empty network is efficient, then the double star will ultimately be efficient if the number of players is increased. So far, we defined the whole set of properties of utility functions which are part of the SCM(W)-class. We also showed how results change under certain circumstances. It is possible to derive many different results. However, next we give a specific example of a very simple network formation model of the SCM(W)-class and derive the set of efficient networks for this utility function. Thereafter, we turn to different network formation classes.

[8]An asymmetric double star is one where the sizes of both stars are different if the number of players is even.

[9]We can also say that the strict ranking of benefits property holds for all all distances except for $d = 2$ and $d = 3$.

Applications of the Classification Scheme

In this section we use the proof above to derive another network formation model from the SCM(W)-class. We call this model the "simple additive symmetric connection model" and it is based on the following utility function

$$u_i = \delta \sum_e \frac{l_e}{e} - c\, l_1 \tag{1}$$

for all $i \in N$ where N is the number of players. δ is the benefit of a link and c is the cost of a link. l_e is the number of links of distance of degree e of player i to any other player in the network. For example, l_1 is the number of direct links of player i. It is easy to check that this utility function satisfies all properties defined above. The following proposition holds.

Proposition 4 *Let all variables be real and continuous then the efficient network is*
(a) the complete network if $c < 0.5\delta$
(b) the star if $0.5\delta < c < (0.5 + 0.25N)\delta$
(c) the empty network if $c > (0.5 + 0.25N)\delta$

Proof: (a) If direct links are more valuable than indirect links then $\delta - c > 0.5\delta$ which implies that the fully connected network g_N is formed if $c < 0.5\delta$. (b) Let us call the upper welfare bound of a connected component with m players and $k \geq m-1$ links $W_U = 2k(\delta - c) + [m(m-1) - 2k]0.5\delta$. The welfare measure of the star is $W_S = 2(m-1)(\delta - c) + 0.5(m-2)(m-1)\delta$. Since $W_S \geq W_U$ for $0.5\delta < c$ the star is efficient among the set of connected components. (c) The star is just restricted by the empty network g_0 which is efficient if $0 > 2(m-1)(\delta - c) + 0.5(m-2)(m-1)\delta$ which can be reduced to $c > \delta(0.5 + 0.25m)$. $\square$

This example illustrates the power of a classification scheme. Given we have specified a utility function such that all properties of a certain class are fulfilled, we can directly use the proof structure of the whole class to derive the set of efficient networks.

3 The MSCM-Class based on the Jefficiency criterion

Here, we define the class of the multiplicative symmetric connection models based on the Nash product which is not the sum but the product of player´s utility functions, i.e. $J = \prod_{i=1}^{N} u_i$. We also call the Nash product 'jefficiency' criterion because it considers both justice and efficiency.

An outcome which maximizes the product of player´s utilities is said to be jefficient.[10] We call this class the MSCM(J)-class. In the theorem below we show that it is closely related to the SCM(W)-class. The new model class is called multiplicative because we claim that utility functions of network formation models of the MSCM-class exhibit not additive separability but multiplicative separability. Hence, we define the following properties in accordance to the SCM(W)-class.

Definition 12 (Multiplicative Separability of Costs)
A utility function exhibits multiplicative separability of costs if the costs of forming links are multiplicatively separable.

Definition 13 (Multiplicative Separability of Benefits)
A utility function exhibits multiplicative separability of benefits if benefits out of direct and indirect links of any degree are multiplicatively separable.

The new set of definitions allows the definition of the MSCM(J)-class.

Definition 14 (MSCM(J)-class)
Let all variables of a network formation model be real and continuous. This model is one out of the MSCM(J)-class if its utility function exhibits the following characteristics:

(i)	∞-Distance Network
(ii)	Anonymity
(iii)	Constant Costs
(iv)	Constant Benefits
(v')	*Multiplicative Separability of Costs*
(vi')	*Multiplicative Separability of Benefits*
(vii)	No Indirect Link-Costs
(viii)	Strict Ranking of Benefits.

[10]See Moebert (2006) for details of the relationship between the efficiency and jefficiency criterion.

In accordance with the results of the SCM(W)-class the following theorem holds in the MSCM(J)-class.

Theorem 2 *If a network is one of the MSCM(J)-class then the empty network, the star network, and the complete network are the only jefficient networks.*

Proof: The only difference between a utility function which is a member of the SCM(W)-class and the MSCM(J)-class is assumption (v') and (vi'). If we logarithmise a utility function of the MSCM(J)-class the utility function turns into one of the SCM(W)-class. If we want to maximize the jefficiency criterion we can also maximize the logarithm of the jefficiency criterion which is $\log J = \sum_{i=1}^{N} \log u_i$. If we plug in the logarithmised utility function of the MSCM(J)-class then the $\log J$ has the same structure as W in the additive SCM(W)-class. Hence, the set of efficient networks coincide in both classes. $\square$

Taking the logarithm of all parameters in the utility function of the SCM(W)-class especially does not hurt the strict ranking of benefits assumption since the logarithm monotonically transforms functions.

4 Rival Network Formation Models

The Co-Author model also introduced by Jackson and Wolinsky (1996) is based on a utility function which exhibits totally different characteristics than the utility functions of the SCM-class. In the co-author model direct links contribute positively and indirect links contribute negatively to the utility functions. Therefore, we can say that the co-author network is not a member of the class of "distance networks". The co-author network instead includes rivalty and we define a "rival network" accordingly.

Definition 15 (d-Rival Network)
A d-rival network is one where direct links contribute positively and indirect links up to distance d contribute negatively to utility functions.[11]

[11]This implies that links of distance $\delta > d$ are not part of any utility function.

That the co-author network is indeed a “2-rival network” is easily seen by considering the utility function of player i

$$u_i(n_i, n_j) = 1 + \left(1 + \frac{1}{n_i}\right)\left(\sum_{j:ij\in g} \frac{1}{n_j}\right) \tag{2}$$

where n_i is the number of links formed by player i, n_j is defined respectively for player j, and $j : ij \in g$ describes the set of direct links formed between player i and any player j in the network g. In this model costs and benefits are not explicitly given. However, we can identify costs and benefits by the following considerations. Increasing the number of links of player i increases and reduces u_i simultaneously. The net effect of both effects might be greater or smaller than zero. u_i is increased since additional players increase the value in the second parenthesis. However, u_i is also decreased since additional players decrease the contribution to i´s utility function of the old players, i.e. the players who have already formed a link. Suppose $n_{\Delta i}$ represent the new links which are formed by player i and $n_{\Delta j}$ represent the number of links the Δj players have after the new links are formed by player i, then the utility function is

$$u_i(n_i, n_{\Delta i}, n_j, n_{\Delta j}) = 1 + \left(1 + \frac{1}{n_i + n_{\Delta i}}\right)\left(\sum_{j:ij\in g} \frac{1}{n_j} + \sum_{\Delta j:ij\in g} \frac{1}{n_{\Delta j}}\right) \tag{3}$$

Then we can define the costs c of the new links as

$$c_i(n_i, n_{\Delta i}, n_j) = \left[\frac{n_{\Delta i}}{n_i(n_i + n_{\Delta i})}\right]\left(\sum_{j:ij\in g} \frac{1}{n_j}\right) \tag{4}$$

and we can also define the benefits b of the new links

$$b_i(n_i, n_{\Delta i}, n_{\Delta j}) = \left(1 + \frac{1}{n_i + n_{\Delta i}}\right)\left(\sum_{\Delta j:ij\in g} \frac{1}{n_{\Delta j}}\right). \tag{5}$$

It is important to keep in mind that $n_{\Delta i}$ is the links added by player i while $n_{\Delta j}$ is the number of links of player j after the link ij has been formed. The formulas derived arrange our thoughts since now we can write.

$$u_i(n_i, n_{\Delta i}, n_j, n_{\Delta j}) = u_i(n_i, n_j) + b_i(n_i, n_{\Delta i}, n_{\Delta j}) - c_i(n_i, n_{\Delta i}, n_j) \tag{6}$$

This formula identifies costs and benefits of player i who consider forming a link. However, the decision to form a link also reduces the utility levels of the n_i players who have already been linked to player i. Therefore, the cost function of these players k is

$$c_k(n_k, n_i, n_{\Delta i}) = \left(1 + \frac{1}{n_k}\right)\left(\frac{n_{\Delta i}}{n_i(n_i + n_{\Delta i})}\right)$$

If a new link is added to a network then the utility function of the players forming the direct link changes and the utilities of all indirectly involved players decreases. Given these cost and benefit functions, it is clear that the co-author model does not exhibit several properties which are satisfied in the SCM-class. The variables influencing the benefit function also affect the cost function. Therefore, "additive separability of costs" and "additive separability of benefits" assumptions are not fulfilled since the cost and benefit functions are nonlinear. Also, the "constant cost" and "constant benefit" assumptions are not not fulfilled. The functions are also so complex that we believe it is not worthwhile defining properties characterizing these functions. However, we continue to show by means of examples that the functions derived are easily applicable.

Example 1 Let us assume the situation described in figure 1 where the number of players is even. In the efficient network, each player forms exactly one link and if the number of players is even the welfare is $U_{eff} = 3N$.[12] If we add an additional link to this network then the reduction in welfare is

$$\Delta U = U_{eff} - U_{new} = 12 - 2\left\{\left[1 + \left(1 + \frac{1}{n_p}\right)\left(\frac{1}{n_c}\right)\right] + \left[1 + \left(1 + \frac{1}{n_c}\right)\left(\frac{1}{n_c} + \frac{1}{n_p}\right)\right]\right\} \quad (8)$$

where N is the number of players, n_p is the number of links formed by the peripheral players, n_c is the number of links of the central players in the line shown in figure 1. The first [.] expresses the utility of one peripheral player and the second [.] expresses the utility of one central player. We know the number of links formed by peripheral players $n_p = 1$ and central players $n_c = 2$. Therefore, $\Delta U = 1.5$. This reduction in welfare can be allocated to the cost and benefit functions above. In our example $n_{\Delta i} = 1$ and $n_{\Delta j} = 2$. Then, the costs $c_p(.) = 1, c_c(.) = .5$, and the benefit of the central player is $b_c(.) = .75$. Summing costs and benefits yields

$$\Delta U = 2\left[c_p(.) + c_c(.) - b_c(.)\right] = 2(1 + .5 - .75) = 1.5 \quad (9)$$

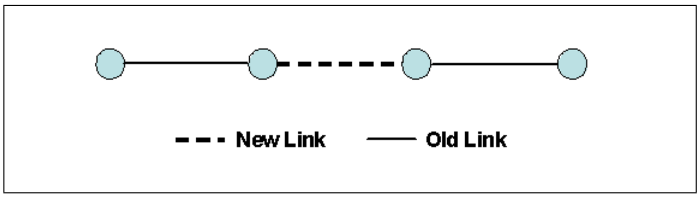

Figure 1: Adding a Link to the Efficient Network

Example 2 Here we extend the example from above and calculate the cost and benefits function if $n_{\Delta i} = 2$. We assume that three player form links to each other as described in

[12]If the number of players is odd the welfare of the efficient network is $U_{eff} = 3(N-3) + 8$ where $(N-3)/2$ components consist of two players and one component consists of three players who form a line.

figure 2. Using the above formula $\Delta U = \frac{10}{3}$ where $U_{eff} = 18$ and $U_{new} = \frac{44}{3}$. The corresponding values of the cost and benefit functions are $b_c(.) = \frac{8}{9}$, $c_c(.) = \frac{6}{9}$, and $c_p(.) = \frac{4}{3}$. The welfare reduction of each pair of peripheral and central player is $\frac{10}{9}$ and the total welfare is decreased by $\frac{10}{3}$.

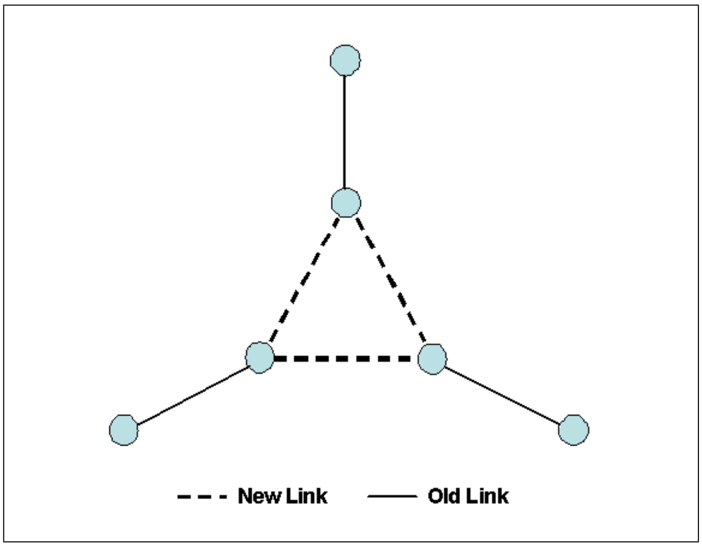

Figure 2: Adding three Links to the Efficient Network

A simple rival network

We haven seen that the cost and benefit functions of the co-author model are quite complicated. If we are interested in modelling much simpler networks exhibiting similar welfare properties then the following model might be appropriate. Let us specify the utility function as

$$u_i = n_i - \beta \sum_{j=1}^{n_i} (n_j - 1) - cn_i \tag{10}$$

where n_i is the number of direct links of player i, n_j is the number of players who have a link to i, c is a cost variable for direct links, and $\beta > 0$ is a weight of the indirect links relative to the direct links. Given our above definitions this utility function has the following properties:
This assumption set is similar to the assumption set of "distance networks" described above, however, here the shortest indirect links reduce utility and can thereby restrict the set of efficient networks to minimal connected networks. If β is close to zero then we are back in a 1-distance network where, depending on the value of c, either the complete or empty network is efficient. If β is high and c is small enough then indirect links are very costly and the efficient network

(i) 2-Rival Network
(ii) Anonymity
(iii) Constant Costs
(iv) Constant Benefits
(v) Additive Separability of Costs
(vi) Additive Separability of Benefits
(vii) No Indirect Link-Costs

is one where if the number of players is even only pairs of components are formed to set the number of indirect links to zero.[13]

In the most interesting cases where c and β are in a medium range it is also possible to characterize the set of efficient networks. This characterization restricts the set of networks which might be efficient to special network structures. A result found by Euler[14] is helpful for this characterization: In a r-regular network $rN = 2L$, where N is the number of players and L is the number of undirected links. Suppose L and N are given, then we can conclude that for some parameter combinations $\frac{2L}{N}$ is unequal a natural number. Otherwise, stated for some values of r, regular networks cannot be formed. Therefore, we define the set of almost regular networks.

Definition 16 (Almost Regular Networks)
The set of almost regular networks is the set of networks such that for a given number of players and for a given number of links $\sum_{i=1}^{N}\sum_{j=1}^{N} \mid n_i - n_j \mid$ *is minimized.*

For regular networks the double sum is zero. Hence, the set of "almost regular networks" includes the set of regular networks. Take into account also that here $L = \frac{\sum_{i=1}^{N} n_i}{2}$. Thus, fixing L and N determines also n_i.

Definition 17 (Cost Class)
Given a utility function which exhibits additive separability of costs, then fixing the number of links L defines a cost-class.

[13]If the number of players is odd, then the efficient network consists of (N-1)/2 components of pairs of players and an empty component. This result is different from the original co-author model (see footnote 12).
[14]In the literature the result is called "Satz von Euler".

This property simplifies the comparison of networks who have the same number of links. In the utility function above this property fixes also the number of direct links. The definitions introduced are useful in the following proposition.

Proposition 5 *Given the utility function above, in each cost-class the efficient network is an almost regular network.*

Proof: The welfare $W = \sum_{i=1}^{N} u_i = \sum_{i=1}^{N} [n_i - \beta \sum_{j=1}^{n_i} (n_j - 1) - cn_i]$. Given a cost-class, i.e. conditioning on the L, we only have to compare the indirect links for each cost class. The maximization of the welfare implies the minimization of the costs of indirect links across all possible networks, i.e. minimizing $\beta \sum_{i=1}^{N} \sum_{j=1}^{n_i} (n_j - 1)$. In a regular network the cost of the indirect links is $\beta N n_i(n_i - 1)$. This utility reduction is minimal for the following reason. Suppose we arbitrarily rearrange the links of the regular network considered, then there might be some players who have less direct links and some players who have more direct links than they had in the regular network. Then, all players who are linked to the players who have less direct links have now less indirect links and welfare is increased. However, all players who are linked to the players who have more direct links have now more indirect links and welfare is decreased. The net effect implies a utility reduction since there are now more players who bear costs out of the more indirect links than there are players who have less costs out of the less direct links. Furthermore, if L and N are such that no regular network can be formed than for the same line of arguments an almost regular network is formed. □

The intuitive idea of the proof is that almost regular networks minimize the number of indirect links in each cost-class. Therefore, the regular networks in rival networks play the opposite role than the star networks in the distance networks. There we were searching for the network structures which maximize the number of indirect links which produce the highest utility.

The overall welfare maximizing network depends on β and c. If the overall welfare maximizing network is a regular network then it satisfies $W(n_i + 1) \leq W(n_i) \geq W(n_i - 1)$. This holds since the linearity of the utility functions guarantees that the utility out of direct and indirect links grows monotonically. If the welfare maximizing network across all cost-classes is not regular then the inequalities above approximate the range of link distributions across all players of the welfare maximizing network. If we plug the welfare functions into the two inequalities above, the following condition is attained: $\frac{1-c}{2\beta} \leq n_i \leq 1 + \frac{1-c}{2\beta}$. For example, if the number of players is even and someone wants to form a welfare maximizing network where each component encloses only two players as in the co-author model then we can set $c \in [1 - 2\beta, 1]$.

5 Conclusion

In this paper we identified properties of utility functions in the network formation literature with respect to the standard welfare criterion. We start the paper by identifying the whole class of symmetric connection models. We used these properties to derive similar network formation models by changing some details of the corresponding utility function. We also investigated the co-author model. This model exhibits several nonlinearities such that the identification of the class encompassing all kind of those models might not improve our understanding of network formation models. Instead, we derived a simpler network formation model where players are also rivals as in the original work. In this network we have shown that the regular networks (and the almost regular networks) determine the set of efficient networks. Given our models we showed that the almost regular networks in the co-author model and the star networks in the class of symmetric connection models can be interpreted as opposite network structures.

The main contribution of this paper is the denomination of certain properties utility functions exhibit. This is very helpful in characterizing the importance of each property of a utility function for the network outcome. We hope that this eventually improves our understanding of more elaborate social network formation models. Thus, this research can be continued by investigating other network formation models with respect to the utilitarian welfare. However, it is also interesting to analyze the properties with respect to other welfare functions. In Section three we defined the class of multiplicative symmetric connection models and showed that this class has similar properties with respect to the Nash welfare criterion than the class of symmetric connection model has with respect to the standard welfare criterion. Of course, this paper can also be extended by evaluating how the utility properties affect the pairwise-stable or pairwise-Nash stable network structures.

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